

EQUIVALENCE OF GEOMETRIC AND COMBINATORIAL DEHN FUNCTIONS

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ABSTRACT. In this paper it is proved that if a finitely presented group acts properly discontinuously, cocompactly and by isometries on a simply connected Riemannian manifold, then the two Dehn functions, of the group and of the manifold, respectively, are equivalent.

1. DEHN FUNCTIONS AND THEIR EQUIVALENCE

Let X be a simply connected 2-complex, and let w be an edge circuit in $X^{(1)}$. If D is a van Kampen diagram for w (see [5]), then the area of D is defined as the number of 2-cells on D , and the area of w , $a(w)$, is defined as the minimum of the areas of all van Kampen diagrams for w . Then the Dehn function of X is defined as

$$\delta_X(n) = \max a(w),$$

where the maximum extends to all loops w of length $l(w) \leq n$.

Given two functions f, g from \mathbb{N} to \mathbb{N} (or, more generally, from \mathbb{R}^+ to \mathbb{R}^+), we say that $f \prec g$ if there exist positive constants A, B, C, D, E such that

$$f(n) \leq Ag(Bn + C) + Dn + E.$$

Two such functions are called equivalent (denoted $f \equiv g$) if $f \prec g$ and $g \prec f$. The importance of Dehn functions is given by the fact that they are invariant by quasi-isometries: when one considers the 1-skeleton of the complex as a metric space with the path metric, where every edge has length one, two complexes with quasi-isometric 1-skeleta have equivalent Dehn functions (see [1]).

Let G be a finitely presented group, and let \mathcal{P} be a finite presentation for G . Let $K = K(\mathcal{P})$ be the 2-complex associated with \mathcal{P} , i.e. the 2-complex with a single vertex, an oriented edge for every generator of \mathcal{P} , and a 2-cell for every relator, attached to the edges according to the spelling of the relator. Then the Dehn function of \mathcal{P} is, by definition, the Dehn function $\delta_{\tilde{K}}$ of the universal covering of K . The fact that two finite presentations \mathcal{P} and \mathcal{Q} for the same group give 2-complexes $\tilde{K}(\mathcal{P})$ and $\tilde{K}(\mathcal{Q})$ with quasi-isometric 1-skeleta, and hence equivalent Dehn functions, leads to the concept of Dehn function of the group G , meaning the equivalence class of the Dehn function of any of its presentations. An extensive treatment of Dehn functions of finitely presented groups is given in [4].

A closely related definition can be formulated in the context of Riemannian manifolds, dating back to the isoperimetric problem for \mathbb{R}^n of Calculus on Variations. Given a Lipschitz loop γ in a simply connected Riemannian manifold M , we call

the area of γ to the minimum of the areas of all Lipschitz discs bounding γ . Then, clearly, we can define the Dehn function of M as

$$\delta_M(x) = \max_{l(\gamma) \leq x} \text{area}(\gamma).$$

It is natural to consider the question of whether the Dehn functions of a simply connected riemannian manifold M and of a finitely presented group G acting properly discontinuously and cocompactly on M agree. The fact that they effectively agree has been implicitly assumed in the literature, while no proof has been given. A closely related statement is given in [2, Theorem 10.3.3], applying to this setting the Deformation Theorem of Geometric Measure Theory ([3, 4.2.9] and [7]), and which provides the basis of the Pushing Lemma below. This paper is devoted to provide a complete and detailed proof of the fact that the two Dehn functions are equivalent. It is known to the author that M. Bridson has an alternate, unpublished proof for the same result. The author would like to thank Professor S. M. Gersten for his encouragement and his useful remarks.

The statement of the theorem is as follows:

Theorem 1.1. *Let M be a simply connected riemannian manifold, and let G be a finitely presented group acting properly discontinuously, cocompactly and by isometries on M . Let τ be a G -invariant triangulation of M . Then the three following Dehn functions are equivalent:*

- (1) *the Dehn function δ_G of any finite presentation of G ,*
- (2) *the Dehn function $\delta_{\tau(2)}$ of the 2-skeleton of τ , and*
- (3) *the Dehn function δ_M of M .*

The fact that δ_G and $\delta_{\tau(2)}$ are equivalent is clear: since G acts cocompactly on τ , there is a quasi-isometry between $\tau^{(1)}$ and the 1-skeleton of $\tilde{K}(\mathcal{P})$ for any presentation \mathcal{P} of G , and the equivalence follows from the results in [1]. We will concentrate on the proof of the equivalence between $\delta_{\tau(2)}$ and δ_M , and the arguments will be mainly geometric, trying to relate the lengths and areas of loops and discs in M with those included in the triangulation τ . The first step in this direction is given by the Pushing Lemma, a complete analog of the Deformation Theorem in Geometric Measure Theory and already stated and proved, in a slightly different way, in [2, Theorem 10.3.3], whose proof we will follow closely.

2. TECHNICAL LEMMAS

The main tool for the proof of the equivalence of the two Dehn functions is the Pushing Lemma, which allows to relate an arbitrary Lipschitz chain in M with another chain which is included in the corresponding skeleton of τ . The fact that we must follow the variation of volume of the chain and prevent its excessive growth is what makes the argument more complicated, since projection from an arbitrary point would lead to arbitrarily large growth of this volume. Techniques from measure theory assure the existence of a center of projection which is far enough from the chain, and hence provides control on the growth of the volume.

Lemma 2.1 (Pushing Lemma). *Let M , G and τ be as above. Then there exist a constant C , depending only on M and τ , with the following property: Let T be a Lipschitz k -chain in M , such that ∂T is included in $\tau^{(k-1)}$. Then there exist*

another Lipschitz k -chain R , with $\partial R = \partial T$, which is included in $\tau^{(k)}$, and a Lipschitz $(k+1)$ -chain S , with $\partial S = T - R$, and such that

$$\text{vol}_k(R) \leq C \text{vol}_k(T) \quad \text{and} \quad \text{vol}_{k+1}(S) \leq C \text{vol}_k(T).$$

The only difference with the statement given in [2] is the fact that their statement for cycles can be extended to chains, since the boundary of the chain is not modified, being included in the $(k-1)$ -skeleton. A statement for cycles is not sufficient, since this lemma will be applied to the discs as well as to the loops, and the fact that $\partial T = \partial R$ is crucial in the proof of the main theorem.

Proof. The proof will proceed by descending induction on the skeleta of τ . So assume that T is included in $\tau^{(i)}$ but not in $\tau^{(i-1)}$, for $i > k$. We want to proceed simplex by simplex, choosing an appropriate point in each simplex and projecting radially from this point the chain T to the boundary of the simplex. The claim we will prove is: there exists a constant C such that for every simplex there exists a point such that projecting T from this point to the boundary of the simplex does not increase the volume of the chain by more than a multiplicative factor C .

To simplify the computations, since the action of the group is cocompact, we can change equivariantly the riemannian metric of M to assume that every simplex is the unit euclidean simplex of side length one. Let σ be an i -simplex of τ , let O be the barycenter of σ , and let r be a positive number such that the ball of center O and radius $3r$ is included in the interior of σ . Let B be the ball of center O and radius r , let $u \in B$, and let B_u be the ball of center u and radius $2r$. Clearly $B \subset B_u$, for all u . Let π_u be the radial projection with center u of $B_u \setminus \{u\}$ onto ∂B_u . Let $Q = T \cap \sigma$. We want to see that there exists a constant v_0 , independent of T and σ , and there exists $u \in B$, clearly dependent on T , with

$$\text{vol}_k(\pi_u Q) \leq v_0 \text{vol}_k(Q).$$

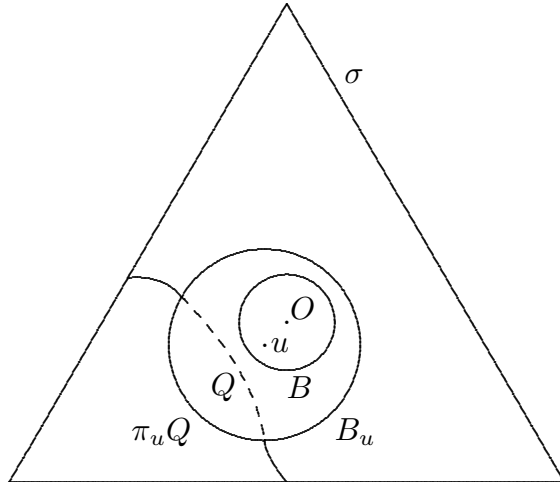


Figure 1: Projecting Q to the boundary of B_u .

For every positive real number v define

$$A_v = \{u \in B \mid \text{vol}_k(\pi_u Q) \geq v \text{vol}_k(Q)\}$$

and let $\alpha(v) = m_i(A_v)$, where m_i is the i -dimensional Lebesgue measure. We want to prove that

$$\lim_{v \rightarrow \infty} \alpha(v) = 0,$$

so it will be enough to choose v_0 such that $\alpha(v_0) < m_i(B)$ to have $B \setminus A_{v_0}$ nonempty.

We have

$$\begin{aligned} \text{vol}_k(\pi_u Q) &\leq \text{vol}_k(\pi_u(Q \cap B_u)) + \text{vol}_k(Q) \\ &\leq \int_{Q \cap B_u} \left(\frac{2r}{\|x - u\|} \right)^k dx + \text{vol}_k(Q), \end{aligned}$$

where the first term accounts for the volume obtained after projecting, and the second term takes care of the possibility of Q and B_u being disjoint. Assume now that $\text{vol}_k Q$ is nonzero (if $\text{vol}_k Q = 0$ then $\text{vol}_k(\pi_u Q) = 0$). Then we have:

$$\begin{aligned} \alpha(v) v \text{vol}_k(Q) &= v \text{vol}_k(Q) \int_{A_v} du = \int_{A_v} v \text{vol}_k(Q) du \\ &\leq \int_{A_v} \text{vol}_k(\pi_u Q) du \leq \int_B \text{vol}_k(\pi_u Q) du \\ &\leq \int_B \left(\int_{Q \cap B_u} \left(\frac{2r}{\|x - u\|} \right)^k dx + \text{vol}_k(Q) \right) du \\ &= (2r)^k \int_{Q \cap B_u} \int_B \|u - x\|^{-k} du dx + \text{vol}_i(B) \text{vol}_k(Q) \\ &\leq (2r)^k \int_{Q \cap B_u} dx \int_{B(O, 3r)} \|u\|^{-k} du + \text{vol}_i(B) \text{vol}_k(Q) \\ &\leq K \text{vol}_k(Q), \end{aligned}$$

where

$$K = (2r)^k \int_{B(O, 3r)} \|u\|^{-k} du + \text{vol}_i(B).$$

Observe that K is finite and independent of T and of σ . The conclusion is that $\alpha(v)v \leq K$. Now, knowing K , we can find v_0 such that $K/v_0 < m_i(B)$, and this v_0 is a constant independent of T and σ . We have found now A_{v_0} with strictly less measure than B , so we can pick a point in $B \setminus A_{v_0}$ from which to project and make sure that the volume increases at most by a multiplicative factor v_0 .

The result of the above argument is the construction of another chain $\pi_u Q$ which is far enough from O . We can now project radially from O to $\partial\sigma$, and the change of volume is bounded since $\pi_u Q$ is at least at a distance r from O . The combination of this change of volume with v_0 gives the constant needed in this precise skeleton. Combining all the constants from all the steps we obtain the desired constant C . Observe that these projections leave $\tau^{(i-1)}$ unchanged, so clearly ∂T is preserved by them.

The $(k+1)$ -chain R is obtained by joining every $x \in Q$ with $\pi_u x$ with a segment. The volume of the piece of R contained in σ is bounded then, as before, by

$$(2r)^{k+1} \int \frac{dx}{\|x - u\|^{k+1}},$$

where the extra factor $2r$ is obtained from the direction of the projection, since each segment has length bounded by $2r$. An argument similar to the previous one shows that projecting from most points in B gives the right bound for the volume. \square

The second lemma states that for a Lipschitz map, almost every point in the target space has a finite number of preimages. It is a direct consequence of the area formula for Lipschitz maps, and it will be used for both loops and discs in the proof of Theorem 1.1.

Lemma 2.2. *Let T be a Lipschitz k -chain in M , where $k \leq \dim M$. Then the set of points in M with infinite preimages under T has Hausdorff k -measure zero.*

Proof. Let σ_k be the standard closed k -simplex, and let

$$E : \sigma_k \longrightarrow M$$

be one of the simplices in T . Since E is a Lipschitz map, by Rademacher's Theorem ([3, 3.1.6]) it is differentiable almost everywhere (with respect to the Lebesgue k -measure), so the Jacobian $J_k E(x)$ is well defined for almost all $x \in \sigma_k$. For $y \in M$, let $N(E, y)$ be the number of elements of $E^{-1}(y)$, possibly infinite, and denote by m_k and h_k the Lebesgue and Hausdorff k -measures, respectively. Then the area formula for Lipschitz maps ([3, 3.2.3]) states that

$$\int_{\sigma_k} |J_k E(x)| dm_k(x) = \int_M N(E, y) dh_k(y).$$

Since E is Lipschitz, we have that $|J_k E(x)|$ is bounded, and since σ_k has finite measure, the integral on the left hand side is finite. So the set where $N(E, y)$ is infinite cannot have positive Hausdorff k -measure, because the right hand side would be infinite. \square

3. PROOF OF THE FIRST INEQUALITY

We will now prove one of the two inequalities involved in the proof of the equivalence of δ_M and $\delta_{\tau(2)}$. Namely,

$$(3.1) \quad \delta_M \prec \delta_{\tau(2)}.$$

Let γ be a Lipschitz loop in M , with length at most n . Using the Pushing Lemma, we can construct a new loop η , of length at most Cn , which is included in the 1-skeleton, and the homotopy between γ and η has area at most Cn .

The loop η is not combinatorial, so we will construct a homotopy between η and a new loop ζ which will be combinatorial, and this homotopy will be included in the 1-skeleton, so it will have area zero. Assume that η is parametrized by

$$\eta : S^1 \longrightarrow M.$$

Since every edge e in $\tau^{(1)}$ has positive Hausdorff 1-measure, by Lemma 2.2 we can choose a point p_e in the interior of e such that the set $\eta^{-1}(p_e)$ is finite. Let

$$\eta^{-1}(\{p_e \mid e \text{ edge of } \tau\}) = \{q_1, \dots, q_r\} \subset S^1$$

with

$$0 \leq \theta_1 < \dots < \theta_m < 2\pi.$$

This gives us a partition of the circle S^1 into arcs $[\theta_i, \theta_{i+1}]$, for $i = 1, \dots, m$ (where $\theta_{m+1} = \theta_1$), such that for every i , one of the two following situations must occur:

- (1) $\eta(\theta_i) = \eta(\theta_{i+1})$, or
- (2) $\eta(\theta_i) = p_{e_i}$ and $\eta(\theta_{i+1}) = p_{e_{i+1}}$ where e_i and e_{i+1} are two edges with a vertex v_i in common.

In the first case, we can collapse $\eta([\theta_i, \theta_{i+1}])$ into $\eta(\theta_i)$, and we can construct a new parametrization of η where for every i , $\eta(\theta_i)$ and $\eta(\theta_{i+1})$ are different. Observe that $\eta([\theta_i, \theta_{i+1}])$ is exactly the concatenation of the two segments $[p_{e_i}, v_i]$ and $[v_i, p_{e_{i+1}}]$, although the map is not a homeomorphism. Find a homotopy between η and the loop ζ such that:

$$\zeta(\theta_i) = \eta(\theta_i),$$

and ζ maps $[\theta_i, \theta_{i+1}]$ homeomorphically to the concatenation of the segments $[p_{e_i}, v_i]$ and $[v_i, p_{e_{i+1}}]$. This homotopy is length-decreasing: call l_i the length of the concatenation of the two segments. Then:

$$\int_{\theta_i}^{\theta_{i+1}} |\eta'(t)| dt \geq \left| \int_{\theta_i}^{\theta_{i+1}} \eta'(t) dt \right| = l_i.$$

So the length of ζ is at most the length of η , and hence at most Cn . Choose $\zeta^{-1}(\tau^{(0)})$ now as the set of vertices of a simplicial structure in S^1 . There is only one situation now which prevents the map ζ from being combinatorial: it is possible that an edge in S^1 is mapped to a loop starting and ending in the same vertex, but since this loop is contained in a single edge of τ , it can be contracted to the vertex. The result of this contraction is now simplicial. Contracting the corresponding edges in S^1 we obtain finally a combinatorial loop in M , of length at most Cn .

This combinatorial loop can be filled combinatorially by at most $\delta_{\tau(2)}(Cn)$ 2-simplices in τ . The conclusion is that

$$\delta_M(n) \leq \delta_{\tau(2)}(Cn) + Cn,$$

and $\delta_M \prec \delta_{\tau(2)}$.

4. CONSTRUCTION OF A SIMPLICIAL DISC

To prove the reverse inequality to (3.1), we start with a combinatorial loop γ in the 1-skeleton of τ , with length at most n . Let

$$f : D^2 \longrightarrow M$$

be a disc in M with boundary γ , and with area a . We want to construct a van Kampen diagram for γ and bound its area in terms of a . The first step is, as before, to use the Pushing Lemma to find a new disc (also denoted f) which is included in $\tau^{(2)}$, and whose area is at most Ca .

Let σ be an open 2-simplex of τ . Again by Lemma 2.2, we can choose a point $p \in \sigma$, such that $f^{-1}(p)$ is finite. Let X be a component of $f^{-1}(\sigma)$. If $X \cap f^{-1}(p) = \emptyset$, then $f|_X$ can be pushed into $\partial\tau$ projecting radially from p , and the resulting map

has area zero. So assume that X is a component of $f^{-1}(\sigma)$ with $X \cap f^{-1}(p) \neq \emptyset$, and note that there are only finitely many of these components. We can also assume that $f|_X$ is surjective, since if it is not we can again project radially from a point not in $f(X)$. Observe that even though these two radial projections can destroy the Lipschitz property for f , from now on we are only going to use the Lipschitz condition on $f|_X$, for those X on which f has not been modified by any radial projection.

Our way to find a lower bound on the area of $f|_X$ will be using the degree of $f|_X$. Since $f|_X$ is differentiable almost everywhere, we can define the degree of $f|_X$ at a point $y \in f(X)$ as

$$\deg f|_X(y) = \sum_{x \in f^{-1}(y)} \text{sign } J_2 f(x)$$

(see [3, 4.1.26]). Moreover, the degree of $f|_X$ is almost constant in $f(X)$, so we can define the degree of $f|_X$ as the value d_X it achieves at almost every $y \in f(X)$. The lower bound is given by the area formula for Lipschitz maps: if u is an integrable function respect to m_2 , we have (see [3, 3.2.3]):

$$\int_X u(x) |J_2 f(x)| dm_2 = \int_{\sigma} \sum_{x \in f^{-1}(y) \cap X} u(x) dh_2,$$

and taking $u(x) = \text{sign } J_2 f(x)$ we obtain:

$$\begin{aligned} \text{area } f|_X &= \int_X |J_2 f(x)| dm_2 \geq \left| \int_X J_2 f(x) dm_2 \right| = \\ &= \left| \int_X \text{sign } J_2 f(x) |J_2 f(x)| dm_2 \right| = \left| \int_{\sigma} \deg f|_X dh_2 \right| = \frac{\sqrt{3}}{2} |d_X|. \end{aligned}$$

Our goal is to find a simplicial map

$$g : D^2 \longrightarrow \tau^{(2)}$$

(with some simplicial structure in D^2) such that in $g|_X$ only $|d_X|$ simplices are mapped by the identity to σ , and the rest of X is mapped to $\partial\sigma$. Then we will have that the combinatorial area of g is bounded the following way:

$$\sum_X |d_X| \leq \sum_X \frac{2}{\sqrt{3}} \text{area } (f|_X) \leq \frac{2}{\sqrt{3}} Ca$$

giving us the required bound. A technical result is needed since g is not combinatorial, but only simplicial, and this result will be the subject of the next section.

The first step in finding the map g is to smooth the map $f|_X$, to be able to use differentiable techniques on it. Let O be the barycenter of σ , and choose $0 < \epsilon < r$ such that:

and let $U_1 = f^{-1}(B(O, r))$ and $U_2 = f^{-1}(B(O, 2r))$. We have that $\overline{U_1} \subset U_2 \subset \overline{U_2} \subset X$. Choose $\delta > 0$ such that $B(x, \delta) \subset X$ for all $x \in U_2$, and such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$, for all $x, y \in X$. Let φ be a C^∞ bump function in \mathbb{R}^2 with support in $B(0, \delta)$, and with integral 1. Then, for $x \in U_2$, we can construct the convolution

$$f * \varphi(x) = \int_{B(x, \delta)} f(x - z) \varphi(z) dz,$$

which is C^∞ in U_2 , and satisfies $|f(x) - f * \varphi(x)| < \epsilon$ for all $x \in U_2$. Also, if $f|_X$ was Lipschitz with constant L , then $f * \varphi$ is also Lipschitz with the same constant: if $x, y \in U_2$,

$$|f * \varphi(x) - f * \varphi(y)| \leq |f(x - z) - f(y - z)| \int_{B(0, \delta)} \varphi(z) dz \leq L|x - y|.$$

Choose now a Lipschitz function α on X with values in $[0, 1]$ and equal to 1 in U_1 and to 0 outside U_2 , and define

$$\tilde{f} = \alpha(f * \varphi) + (1 - \alpha)f|_X.$$

Note that \tilde{f} is defined only on X . Then \tilde{f} satisfies the following properties:

- (1) $|f(x) - \tilde{f}(x)| < \epsilon$ for all $x \in X$,
- (2) \tilde{f} is smooth in U_1 ,
- (3) $\tilde{f} = f$ in $X \setminus U_2$,
- (4) \tilde{f} is Lipschitz, and
- (5) $\deg \tilde{f} = \deg f|_X$.

The first three properties are clear from the construction, and property (4) holds because $f|_X$, $f * \varphi$ and α are all Lipschitz. To see that the degree is unchanged, since the degree is almost constant, and $f|_X$ and \tilde{f} agree outside U_2 , we only need to choose a point in $\sigma \setminus B(O, 2r + \epsilon)$ for which the degree is d_X for both $f|_X$ and \tilde{f} .

We can now use Sard's Theorem ([6]) to claim the existence of a regular value for \tilde{f} in $B(O, r - \epsilon)$ whose preimages are all in U_1 . Let q be this regular value and let p_1, \dots, p_m be its preimages. Let V be an open disc with center q such that $\tilde{f}^{-1}(V) = V_1 \cup \dots \cup V_m$, where the V_i are discs around p_i , pairwise disjoint, and such that $\tilde{f}|_{V_i}$ is a diffeomorphism. In general, we will have that $m > |d_X|$, for which we will have to cancel discs with opposite orientations. Assume V_{m-1} and V_m are mapped to V with opposite orientations. Choose $a \in \partial V_{m-1}$ and $a' \in \partial V_m$ with $\tilde{f}(a) = \tilde{f}(a')$, and join a and a' with a simple path λ such that $\tilde{f}(\lambda)$ is nullhomotopic in $\sigma \setminus V$, which can be done because the map

$$\tilde{f} : X \setminus \bigcup_{i=1}^m V_i \longrightarrow \sigma \setminus V$$

induces a surjective homomorphism of the fundamental groups. Contracting $\tilde{f}(\lambda)$ we can assume $\tilde{f}(\lambda)$ is the constant path $\tilde{f}(a)$. Remove the discs V_{m-1} and V_m and perform surgery along λ . The new boundary thus created is mapped to ∂V

under \tilde{f} by a map from S^1 to itself of degree zero. Extend this map to a map from D^2 to S^1 and attach it to \tilde{f} along this boundary. For the new map (which we will continue calling \tilde{f}), the preimage of q consists only of the points p_1, \dots, p_{m-2} . Repeating this process we will obtain a map where now only the discs $V_1, \dots, V_{|d_X|}$ are mapped to V , and all with the same orientation.

Choose (temporarily) a sufficiently fine subdivision of τ such that there is a 2-simplex W in V , and let $\rho_i = \tilde{f}^{-1}(W)$. Modify the map in X by composing with the expansion of W into all σ .

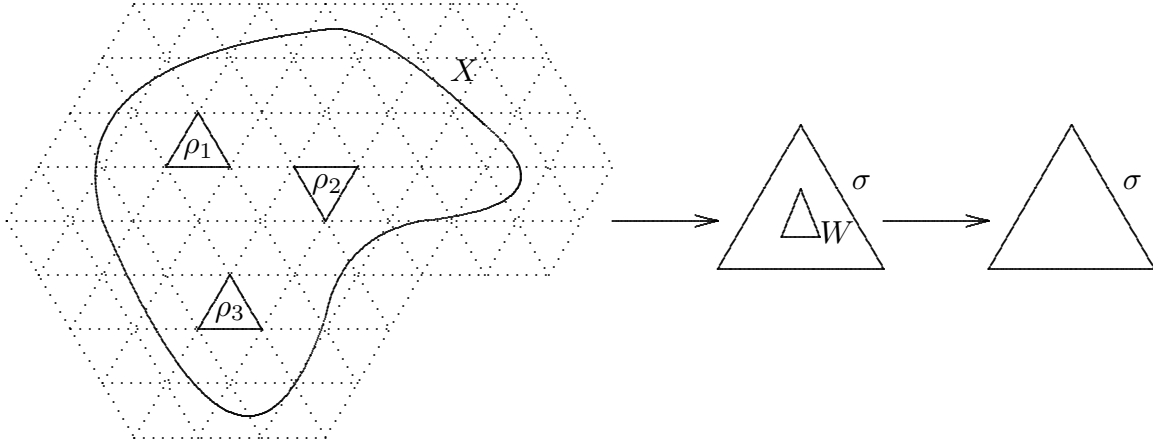


Figure 2: *Making the map f simplicial*

After this process is done for all σ , we obtain a map from D^2 to $\tau^{(2)}$, where all the ρ_i are sent homeomorphically to 2-simplices of τ , and the rest is sent to the 1-skeleton of τ . To finish the construction of g , find a simplicial structure on D^2 compatible with the simplicial structure on the original loop γ and which includes all the ρ_i obtained for all σ as 2-simplices, and approximate simplicially within $\tau^{(1)}$ the map \tilde{f} relatively to all the ρ_i and to γ . The result is now simplicial, and the number of simplices sent by g homeomorphically to 2-simplices in τ is

$$\sum_X |d_X| \leq \frac{2}{\sqrt{3}} Ca.$$

This map is not a van Kampen diagram yet, since it is only simplicial, and that is the subject of the next section.

5. DEGENERATE DEHN FUNCTIONS

Recall from the definition of van Kampen diagram that the map is required to be combinatorial, i.e. every open cell is mapped homeomorphically onto an open cell of the target. It would be useful to extend this definition to maps which are only simplicial, as the one obtained in the previous section. This leads to the following definitions:

Definition 5.1: Let K be a simplicial 2-complex, and let w be a simplicial loop in $K^{(1)}$. A degenerate van Kampen diagram for w is a simplicial map from a planar contractible 2-complex D , with some simplicial structure, to K , such that the map restricted to the ∂D is w . The length of w is defined as the number of 1-simplices on S^1 which are mapped homeomorphically to 1-simplices of K , and similarly the area

of the degenerate van Kampen diagram is the number of 2-simplices of D which are mapped homeomorphically to 2-simplices of K . Then, given a path w , its area is defined as the minimum of the areas of all degenerate van Kampen diagrams for w . And the degenerate Dehn function of K is defined as:

$$\delta_K^{\text{deg}}(n) = \max_{l(w) \leq n} \text{area}(w).$$

In the context of the previous section, we have proved the inequality

$$\delta_{\tau(2)}^{\text{deg}} \prec \delta_M,$$

since the map g constructed in section 4 is a degenerate van Kampen diagram. The result that finishes the proof of the main theorem is the following:

Theorem 5.2. *Let K be a simplicial 2-complex. Then,*

$$\delta_K \equiv \delta_K^{\text{deg}}.$$

Proof. One of the inequalities is obvious: let w be a simplicial loop in K . Modify the simplicial structure on S^1 , taking any 1-simplex which maps into a vertex and collapse it. This produces a combinatorial loop, which can be filled with a combinatorial disc, which is in particular simplicial. Then,

$$\delta_K^{\text{deg}} \prec \delta_K.$$

For the opposite inequality, let w be now a combinatorial loop. We can fill it with a simplicial map f from the disc to K , and the only thing we need to do is construct a combinatorial disc with smaller area. Choose a vertex of K , and let L be a connected component of $f^{-1}(v)$. Change the disc D^2 by

- (1) collapsing L to a point, and
- (2) every 2-simplex with a face adjacent to L , but not in L , has to be sent to an edge of K adjacent to v . Collapse all these simplices to edges.

Lemma 5.3. *The result of the collapsing indicated above is a planar contractible simplicial complex with some 2-spheres attached to a vertex.*

Proof. Attach a 2-cell e to the boundary of D^2 to obtain a finite cellular structure on S^2 . Assume L is contractible. Then clearly the result of the collapsing will be a 2-manifold since every edge is still adjacent to only two faces, and the star of every vertex (including the vertex obtained in the collapsing) is an open disc. An easy count of vertices, edges and faces gives the Euler characteristic equal to 2, and hence the result is a 2-sphere. Eliminating the cell e we obtain a planar contractible simplicial complex.

If L is not contractible, then the same argument can be applied to every connected component of $S^2 \setminus L$, obtaining a wedge of 2-spheres, and only one of them contains the cell e . Again eliminating e we obtain a planar contractible complex with some spheres attached to a vertex. \square

Excising the spheres we obtain a new degenerate van Kampen diagram, and since w is a combinatorial map, no edge of the boundary has been collapsed, and

then the map on the boundary is still w . Doing this for all connected components of $f^{-1}(v)$, and for every v , we obtain a new map from some planar contractible 2-complex into K which is now combinatorial, i.e. a (nondegenerate) van Kampen diagram. Observe that this process cannot increase the area, but can only decrease it when the 2-spheres are cut off. Then we produced a van Kampen diagram for w with smaller area than the original degenerate van Kampen diagram. This proves the inequality

$$\delta_K \prec \delta_K^{\text{deg}}.$$

□

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